[redacted]

**UNPUBLISHED PRELIMINARY DATA**

# D Region Probe Theory


By

David P. Hoult

The Pennsylvania State University
Ionosphere Research Laboratory
and
Department of Aeronautical Engineering
University Park, Pennsylvania



## Abstract

A systematic theory of a parachute borne blunt probe operating in an altitude range from 50 km to 80 km is presented. The relationships between positive ion, negative ion, and electron densities and the current to the probe and the probe voltage are found. It is shown that order of magnitude errors occur if a free molecular flow theory is used in this altitude range. No direct measurement of negative ion density is available for a probe voltage large compared to kT/e, the case considered. The results are somewhat analogous to Gerdien condenser theory.




[redacted]

I. Introduction

This paper presents the essential physics of the operation of a probe used to measure the number density of charged particles in the D region. The nose of a parachute borne blunt probe is charged to a potential and the current to a known area of the nose is measured. The probe is parachuted to produce subsonic flow and hence avoid the significant errors which occur in supersonic flow [Hoult, 1964]. The significance of such an experiment is first, that positive and negative ions are not observable by ground based techniques, and second, the experimental determination of the number densities of positive ions, negative ions, and electrons is a necessary first step in understanding the complex chemical structure of the D region. The relationship between current, voltage, and the number densities of ions and electrons is required to interpret such data. Unfortunately, there is no existing probe theory for this purpose. For example, Lam [1964] assumes that the ratio of Debye length to body dimension is small, and ignores negative ions. The converse distinguishes the operation of a probe in the D region.

A typical probe dimension is 10 cm; velocity, 100 m/s; and voltage, 5 v. This gives a Mach number of about 3/10, and the ratio of voltage to $\frac{kT}{e}$ (k = Boltzman's const, T = temp, e = electron charge) is about $2 \times 10^2$. At 50 km, the number density of neutrals, $n(A_2)$ ($A_2$ stands for the chemically similar molecules NO, $N_2$ and $O_2$) is $10^{16}/cm^3$ at 80 km. The mean free path for neutral particles, $\Lambda$, is $10^{-2}$ cm at 50 km 1 cm at 80 km. Hence at 80 km the ratio of mean free path to body dimension is 1/10, which is about the

[redacted]

limit of a continuum theory. The number density of positive ions, $n(A_2^+)$ is about $10^3/cm^3$ and roughly constant with altitude. The number density of electrons, n(e) is about $10^3$ at 80 km and $10^2/cm^2$ at 50 km. The gas is neutral so negative ions (here assumed to be $A_2^-$) make up the difference in charge density. The Debye length for electrons is 10 cm at 50 km and 3 cm at 80 km. Hence the ratio of Debye length to body dimension is 1 at 50 km; at 80km this ratio is about 1/3. Due to the small concentration of charged particles in the flow, the fluid mechanical motion is unchanged by the ion collection process. For an incompressible flow (Mach number <<1) the Reynolds number, Re, (= $UL/\nu$, U = free stream velocity, L = probe dimension, $\nu$ = kinematic viscosity) is the important parameter. Re varies from $5 \times 10^2$ at 50 km to 10 at 80 km. Hence there is a laminar boundary layer over the probe.

The ratio of the change in velocity of a charged particle of mass m, accelerated for one mean free path by an electric field E due to a voltage on the probe of V, to the thermal speed $v_o$ in the gas is

$$\frac{\Delta v}{v_o} \approx \frac{eE}{m} \left(\frac{\Lambda}{v_o}\right) \frac{1}{v_o} . \tag{1}$$

Using $v_o \sim \sqrt{\frac{kT}{m}}$ and estimating $E \sim {}^V/L$ gives

$$\frac{\Delta v}{v_o} \sim \left(\frac{eV}{kT}\right) \frac{\Lambda}{L} . \tag{2}$$

If mobility is to have meaning, this ratio must be as small as say $10^{-1}$.

This gives an upper bound on the probe voltage of $2 \times 10^{-2}$ volts at 80 km and 2 volts at 50 km. However, data on mobility [ von Engel, 1955, fig. 61 ] in air show that this estimate is conservative by nearly a factor of $10^2$, perhaps due to clustering. Thus the concept of mobility is appropriate up to altitudes near 80 km. By the principle of detailed balancing, the concept of diffusion is appropriate up to about 80 km also. The diffusion Reynolds number, Rd(= UL/D, where D = diffusion coefficient for positive or negative ions) is $10^3$ at 50 km and 10 at 80 km.

In the next Section (Sec. II) the nondimensional equations and boundary conditions are written down. It is shown that the density is so small, and the probe voltage so high ($\sim$ 5 v) that the electric field is everywhere unchanged by the presence of the charged particles. Thus the electric field is that of the probe in a vacuum. Further, the region where positive ion diffusion is important, for a negatively charged probe, is much thinner than the thickness of the viscous boundary layer. Thus it happens that the positive ion current to a negative probe depends only on the electric field at the surface of the probe. For a positive probe, the same is true of negative ion and electron current. For a positive probe, the electron current always dominates the ion current by a factor $\beta$, the ratio of ion diffusion coefficient to electron diffusion coefficient. The effects of flow occur principally in determining the negative ion current to a negative probe. This current is very small compared to the positive ion current to a negative probe.

In the final section (Sec III) a sample calculation is presented to make explicit the general theory of section II. The flow over a disk, normal to the free stream, is considered. The disk is charged to a fixed voltage and the current to a small area at the center of the disk is computed. The relationship between current, voltage and number densities is found.

## II. The Governing Equations

The first approximation to subsonic flow is the incompressible case; the first correction is O(Mach number)$^2$. The incompressible approximation greatly simplifies the theory as the variation in mobility and diffusivity, and the effects of chemical reactions are all O(Mach number)$^2$. Due to the small concentration of charged particles, the fluid mechanical motion is unchanged by the ion collection process. Once the geometry of the probe, the Mach number ($<<1$), and the Reynolds number, Re, are given, t he flow velocity is (in principal) known.

Non dimensionalize the flow velocity with U, the coordinates (x, y, z) with L, the potential with $\frac{e}{kT}$, and the charge density with the number density of electrons far from the body. Denote the nondimensional quantities as follows: $\vec{q}$ is the flow velocity, $\phi$ the potential, $n^+$ the positive ion density, $n^-$ the negative ion density, and $n^e$ is the electron density. Using this notation, and Einstein's relation, the equations are [Lam, 1964]

$$\alpha^2 \nabla^2 \phi = n^+ - n^- - n^e , \tag{1}$$

$$\nabla (Rd\, n^+ \vec{q} - n^+ \nabla \phi - \nabla n^+) = 0, \tag{2}$$

$$\nabla(Rd\, n^- \vec{q} + n^- \nabla \phi - \nabla n^-) = 0, \tag{3}$$

$$\nabla(\beta\ Rd\, n^e \vec{q} + n^e \nabla \phi - \nabla n^e) = 0, \tag{4}$$

where $\alpha$ is the ratio of Debye length to body dimension, and $\beta$ is the ratio of ion to electron diffusion coefficients and is approximately the square root of the electron to ion mass ratio. The temperature of all charged particles is equal to the temperature of the neutrals (a good assumption in the altitude range under consideration).

Let $\lambda$ be the ratio of negative ion density to electron density. Then the boundary conditions on charge density and potential far from the body are

$$n^+ = 1 + \lambda \tag{5}$$

$$n^- = \lambda \tag{6}$$

$$n^e = 1 \tag{7}$$

$$\phi = 0 \tag{8}$$

For an absorbing body, the number density of charged particles at the wall is zero. At the wall, the potential is known. Hence the boundary conditions at the wall read

$$n^+ = 0 \tag{9}$$

$$n^{-} = 0 \tag{10}$$

$$n^{e} = 0 \tag{11}$$

$$\phi = \phi_w \tag{12}$$

Since $\phi_w$ (= $\frac{eV}{kT}$ , V = probe potential) is large, set

$$\phi = \phi_w \, \varphi(x, y, z) \tag{13}$$

Substituting 13 into 1 shows that the right hand side of 1 is at most $O(1/\alpha^2\phi_w)$. Hence

$$\nabla^2 \varphi = 0 \tag{14}$$

is a first approximation to $\phi$. Thus the electric field is that of a probe in vacuum. This is due to the very low number density of charged particles ( $\alpha \sim O(1)$ ) and the high wall potential ($\phi_w >> 1$).

Consider a negative probe ($\phi_w < 0$). Let z be the normal to the wall . Using 14, and the fact that the flow is incompressible ($\nabla . \vec{q} = 0$), equation 2 becomes

$$(Rd\,\vec{q} - \phi_w \vec{\nabla} \varphi) . \nabla n^{+} - \nabla^2 n^{+} = 0 \tag{15}$$

For diffusion to be as important as convection, the last term in 15 must be as large as the effects of mobility ($-\phi_w \nabla \varphi . \nabla n^{+}$) . Following the standard practice of boundary layer theory, set

$$\tilde{z} = \phi_w z. \tag{16}$$

Now if $\tilde{z}$ is of order one, the convection term in 15 is negligible, as the viscous boundary layer is $O(1/\sqrt{Re})$ thick which is $>> O(1/\phi_w)$, and as the velocity $\vec{q}$ is zero at the wall. Let the electric field at the wall be

$$\frac{\partial \phi}{\partial z} \doteq - \phi_w f(x.y) \tag{17}$$

Then, for $\tilde{z}$ of order one, equation 15 becomes

$$- f(x,y) \frac{\partial n^+}{\partial \tilde{z}} + \frac{\partial^2 n^+}{\partial \tilde{z}^2} = O \tag{18}$$

The appropriate solution of 18 is

$$n^+ = (1 + \lambda)(1 - \exp + f(x, y) \phi_w z) \tag{19}$$

For a negative probe, $\phi_w < 0$. Thus 19 has the following properties: first, in the ion diffusion region, which is $O(\frac{1}{\phi_w})$ thick, 19 satisfies 18, the approximate form of 15. Second, outside of the ion diffusion region 19 satisfies 15, as $n^+$ is a constant for $\tilde{z}$ large. Thus 19 is the first approximation to $n^+$ everywhere.

In dimensional terms, the ion current to an area element dS of the probe surface is

$$dI = e D \left. \frac{\partial n (A_2^+)}{\partial Z} \right|_{wall} dS \qquad (20)$$

Using 19, this becomes

$$dI = e n_\infty (A_2^+) U d S \left( \frac{-\phi_w}{Rd} \right) f \left( \frac{X}{L} , \frac{Y}{L} \right) \qquad (21)$$

Thus the positive ion current to a negative probe is linear in potential.

For a positive probe, the electron density and negative ion density are found, by the same argument to be (note that $\phi_w > 0$)

$$n^e = 1 - \exp -f(x,y) \phi_w z \qquad (22)$$

$$n^- = \lambda (1 - \exp - f(x,y) \phi_w z) \qquad (23)$$

The electron and negative ion current to an area dS of a positive probe are

$$dI = e n_\infty (A_2^-) U dS \frac{\phi_w}{Rd} f \left( \frac{X}{L}, \frac{Y}{L} \right)$$

$$+ e n_\infty (e) U d S \frac{\phi_w}{\beta Rd} f \left( \frac{X}{L}, \frac{Y}{L} \right) \qquad (24)$$

Again this current is linear in potential. However, as $\beta \approx 5 \times 10^{-3}$ (approximately the square root of electron to ion mass ratio), the electron current always dominates the negative ion current. Hence it is impossible to measure negative ion current. Hence it is impossible to measure negative ion densities with a positive probe.

Note that use of a naive theory based on free molecular flow (current ~ area x velocity x number density x charge), can be in error, from the data in the introduction, by a factor of 10 for positive ions and $10^2$ for electrons.

The theory for positive probes is complete with equation 24, as the electron current completely dominates all other effects. For negative probes, the electron and negative ion densities must be discussed. Referring to equations 3, 4, 13, and 14, it is seen that the effect of convection on electrons, $O(\beta Rd)$, is negligible compared to the effect of mobility $O(\phi_w)$ except very far from the probe. Electrons are stopped far from the probe. Hence the electron current to a negative probe is negligible. However the effects of convection on negative ions $O(Rd)$ can be larger than the effect of mobility $O(\phi_w)$. Hence negative ions are stopped near the probe, and the negative ion current may not be completely negligible. Explicit formulae for negative ion and electron densities, which depend on the detailed structure of the flow field, are worked out for the case of a disk in the next section. It is shown that the negative ion current is in fact very small.

## III. A Charged Disk Normal to the Stream

Consider a disk of radius L normal to the free stream velocity. The flow is incompressible and axisymmetric. Let z be the (non-dimensional) direction normal to the disk, and the r the cylinder radius. The current collected inside $r = r_o << 1$ is measured. The relationship between the current to the collecting area, the voltage applied and the number density of the media is to be found. Re, Rd, and $\phi_w$ are given. Hence the velocity field and potential field around the probe are known.

Define a stream function in terms of the fluid mechanical velocities as

$$\frac{\partial \psi}{\partial r} = - r q_z , \quad \frac{\partial \psi}{\partial z} = r q_r \tag{25}$$

As the collector is small ($r_o << 1$), only the velocity field and potential in the region near the axis ( $r << 1$) are required. Outside the boundary layer, the stream function [Milne - Thompson, 1960] is, for $r << 1$,

$$\psi = + \frac{1}{2} r^2 + \frac{1}{\pi} (z - (1 + z^2) \cot^{-1} z) \left( \frac{r^2}{1 + z^2} \right). \tag{26}$$

The potential of a disk for $r << 1$ is [Jeans, 1925 ]

$$\phi = \phi_w \frac{2}{\pi} \tan^{-1} \left( \frac{1}{z} \right). \tag{27}$$

Using 17 and 27 thus gives, for $r << 1$,

$$f = \frac{2}{\pi} \tag{28}$$

By 21 and 28, the positive ion current to the collecting area of a negative disk is

$$I_+ = e\, n_\infty (A_2^{+})\, U (\pi\, r_o^{\,2}\, L^2)\, \frac{2}{\pi}\, \frac{\phi_w}{Rd} \tag{29}$$

Likewise, the total current to the collecting area of a positive disk is approximately (the correction is O( β )

$$I_e = e\, n_\infty (e)\, U (\pi\, r_o^{\,2}\, L^2)\, \frac{2}{\pi}\, \frac{\phi_w}{\beta Rd} \tag{30}$$

Now consider the negative ion density, and electron density near a negative disk. The equations to be solved are 3 and 4 which take the form

$$(Rd\, q + \phi_w\, \vec{\nabla} \varphi) \cdot \nabla n^- - \nabla^2 n^- = O \tag{31}$$

$$(\beta\, Rd\, \vec{q} + \phi_w\, \vec{\nabla} \varphi) \cdot \nabla n^e - \nabla^2 n^e = O \tag{32}$$

Near the axis the electric field and flow velocity in the radial direction are O(r). The z components of electric field and flow velocity have corrections which are O(r). Taking these facts into account, using the equations (31, 32) and the boundary conditions (6, 7, 10, 11), it can be shown that near the axis the number densities behave as

$$n^- = n^-(z) + O(r^3) \tag{33}$$

$$n^e = n^e(z) + O(r^3). \tag{34}$$

Hence 33 and 34 reduce to

$$\left(Rd\, q_z + \phi_w \frac{\partial \varphi}{\partial z}\right) \frac{dn^-}{dz} - \frac{d^2 n^-}{dz^2} = 0, \tag{35}$$

$$\left(\beta\, Rd\, q_z + \phi_w \frac{\partial \varphi}{\partial z}\right) \frac{dn^e}{dz} - \frac{d^2 n^e}{dz^2} = 0, \tag{36}$$

the error being $O(r)$. Define to be $\Phi$ to be

$$\Phi = \int_0^z q_z \, d\, z. \tag{37}$$

Outside the boundary layer, $\Phi$ is the velocity potential. The solution of 35 and 36 satisfying the boundary conditions is

$$n^- = \frac{\lambda \int_0^z \exp\left(Rd\,\Phi + \phi_w \varphi\right) dz}{\int_0^\infty \exp\left(Rd\,\Phi + \phi_w \varphi\right) dz} \tag{38}$$

$$n^e = \frac{\int_0^z \exp\left(\beta Rd\,\Phi + \phi_w \varphi\right) dz}{\int_0^\infty \exp\left(\beta\, Rd\,\Phi + \phi_w \varphi\right) dz} \tag{39}$$

The corresponding negative ion and electron currents to the wall are in dimensional terms,

$$I_- = e \frac{n\,(A_2^-)\, U\,(\pi\, r_0^2\, L^2)}{Rd} \left[\frac{e^{+\phi_w}}{\int_0^\infty \exp\left(Rd\,\Phi + \phi_w \varphi\right) dz}\right], \tag{40}$$

$$I_e = \frac{e\, n_\infty(e)\, U\, (\pi\, r_o^2\, L^2)}{Rd} \left[ \frac{e^{+\phi_w}}{\int_o^\infty \exp\,(\beta Rd\ \Phi + \phi_w \varphi)\, dz} \right]. \tag{41}$$

For a negative disk, $\phi_w < 0$. Evidently the size of the currents is determined by evaluating the terms in brackets in 40 and 41. This is easily done by the method of steepest descent. [ See Jeffreys and Jeffreys, 1956 ] . The results for the electron current are particularly simple. The integral in 41 can be approximated by an integral near the saddel point, which is found by setting the derivative of the exponent equal to zero. This gives, if the saddel point is located at $z = z_o$,

$$\beta Rd q_z(z_o) + \phi_w \frac{\partial \varphi}{\partial z}(z_o) = 0. \tag{42}$$

This equation locates the distance from the probe, $z_o$, at which the velocity due to mobility is equal and opposite to the flow velocity. This "stand off" distance, for electrons, is large:

$$z_o \cong \sqrt{-\phi_w/\beta Rd}. \tag{43}$$

By using 26, 27, 43 and the method of steepest descent, it can be shown that the electron current to the probe is of order

$$\exp\left\{ -\left|\phi_w\right| + O\left(\sqrt{-\phi_w/\beta Rd}\right)\right\} \tag{44}$$

and hence is completely negligible. Turning to the case for negative ions, one can say immediately that if the stand off distance is large, or even of order one, that the negative ion current is negligible. The only interesting case is to consider the stand off distance less than order one, but greater than the boundary layer thickness. In this case

$$z_o = \frac{-\phi_w}{Rd} \tag{45}$$

From the introduction, $z_o$ is always greater than $O\left(\frac{1}{\sqrt{Re}}\right)$, which is the boundary layer thickness. Referring to equation 37, it is seen that the requirement that $z_o < 1$ means that convection dominates mobility effects until a region close to the wall is reached (the velocity is zero on the wall). Even in this case the negative ion current turns out to be very small compared to the positive ion current.

$$I_- = \frac{e\, n_\infty (A_2^-)\, U\, (\pi\ r_o^2\ L^2)}{Rd^{3/2}}\ \pi\ e^{-\frac{2}{\pi}\ \phi^2{}_w/Rd} \tag{46}$$

In closing, note that currents of the form given by 29 or 30, by Einstein's relation, may be written in a form analogous to Gerdien condenser theory as

$$I \sim e\, n_\infty \text{ Area x mobility x electric field at wall.} \tag{47}$$

Acknowledgments

The author wishes to thank Dr. L. C. Hale and Mr. R. Willis for many helpful discussions. The help of the Ionosphere Research Laboratory of Pennsylvania State University is appreciated.

This work was supported in part by Grant DA-ARO-D-31-124-G537 and in part by NASA Grant NsG-134-61.

## References

Hoult, D. P., 1964, Journal Geophys. Res. 69, 4617.

Jeans, 1925 The Math Theory of Electricity and Magnetism, Cambridge University Press, P. 249.

Jefferies, H. and Jefferies, B. S., 1956, Methods of Mathematical Physics, Cambridge University Press.

Lam, S. C. (1964) AIAA Journal 2, 256.

Milne-Thompson, L. M., 1960, Theoretical Hydrodynamics, MacMillian Company, New York, P. 476.

von Engle, A., 1955, Ionized Gases, Oxford at the Clarendon Press.